The Analogy of the Theory of Kummer's Ideal Numbers with Chemistry and its Prototype in Plato's Concept of Idea and Number

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material before 1500, and the other on such material produced since 1900 probably with other important memoirs omitted from the present volume through lack of space.

No selections have as yet been made, either of topics or contributors, and it is hoped that volunteers will notify the Committee as soon as possible of any article or articles that they will be able to prepare.

It is desirable to bring this matter to the attention of mathematical clubs of various kinds in the graduate schools of our universities, thus enlisting the interest of the advanced students in mathematical courses. It is probable that a considerable number of the articles will be written by men and women of this type.

Suggestions of topics and, what is even more important, offers to edit certain ones in which the volunteer is especially interested, may be sent to Professors R. C. Archibald, Brown University, Providence, R. I.; Florian Cajori, Berkeley, Calif.; or David Eugene Smith, 501 West 120th Street, New York City.

A few possible topics are the following: decimal fractions; Delamain and Oughtred on the slide rule; the binomial theorem; mathematical induction; Cardan's solution of the cubic and biquadratic (Del Ferro, Tartaglia, Ferrari); De Moivre's theorem $\cos nx+i \sin nx=(\cos x+i \sin x)^n$ and the relation of Cotes, $ix=\log(\cos x+i \sin x)$; definitions of e; uniform convergence; Determinants; Hill-Poincaré infinite determinants; Kummer's ideals; Descartes, selections from *La Géométrie;* Desargues's theorem on perspective triangles; Gauss's statement on the construction of regular polygons; Bolyai's first essay (1823); Lobachevsky's *Pan-Geometry* (1835); circular points at infinity; Gauss's measure of curvature of surfaces; Bolzano (Weierstrass) nowhere differentiable continuous functions; Bessel's functions and numbers.

THE ANALOGY OF THE THEORY OF KUMMER'S IDEAL NUMBERS WITH CHEMISTRY AND ITS PROTOTYPE IN PLATO'S CONCEPT OF IDEA AND NUMBER[1]

By HARRIS HANCOCK, University of Cincinnati

As this article is intended for those who are interested in philosophic as well as mathematical problems, it may be well to indicate the nature of Kummer's ideal numbers by means of two examples. The first is due to Sommer, the second to Hensel. The presentation of the ideal numbers which is given immediately after these examples is not necessary for a casual survey of the subject under consideration. The mathematician, however, who reads the entire article, will, I believe, get a deeper insight into the Plato concept.

[1] Read before the Ohio Section of the Mathematical Association of America, April 5, 1928.

The following is the example used by Sommer (*Vorlesungen über Zahlentheorie*). Consider as a fixed realm of rationality the realm composed of integers of the form $4n+1$ and permit in the discussion only the operations of multiplication and division in their usual sense. In the series

$$1,5,9,13,17,21, \cdots, \quad 45, \cdots, \quad 117, \cdots, \quad 517, \cdots$$

it is clear that the product of any two integers of the series is an integer of the series since $(4m+1)$ $(4n+1)$ is of the form $4g+1$, where m and n are any two positive integers and $g=4mn+m+n$. The numbers 5, 9, 13, 17, 21, 29, $\cdots$, are irreducible, that is, they take the place of prime integers in this realm, in that they are not equal to the product of any other two integers of the realm.

Next observe that the number 10857 may be factored in the following two different ways $10857=141\cdot 77=21\cdot 517$, where 21, 77, 141 and 517 are prime numbers in the fixed realm. Thus it is seen that in this realm the factorization of an integer into its prime factors is not a unique process. This factorization, as is well known, becomes unique, if the fixed realm in question is extended so as to comprise what is known as the realm of *all* rational integers, in which $10857=3\cdot 7\cdot 11\cdot 47$.

Kummer's thought when applied to the above special case, consists in replacing the factors 3, 7, 11, 47, by what may be called "ideal numbers" in the restricted realm. In this realm, observe that the integers 3, 7, 11, 47 as such do *not* exist. To grasp the import of the "ideal numbers," denote the greatest common divisor of two integers a and b by the symbol (a, b) and observe that $(a, b)=(b, a)$. In the extended realm (that is, the realm of all natural integers) 3 is the greatest common divisor of 21 and 141 and may be written $3=(21, 141)$, where 21 and 141 are entities in the restricted realm. Further, we may put

$$7 = (21,77)\ ; \qquad 11 = (517,77)\ ; \qquad 47 = (517,141)\ ;$$

$$(141) = (141,21)(141,517)\ ; \quad 77 = (77,21)(77,517)\ ;$$

and

$$(10857) = (141,21)(141,517)(77,21)(77,517) = 141 \cdot 77.$$

On the other hand,

$$(21) = (21,141)(21,77)\ ; \ (517) = (517,77)(517,141)\ ;$$

and

$$(10857) = (21,141)(21,77)(517,77)(517,141) = 21 \cdot 517.$$

In both cases the factorization of 10857 through its "ideal" prime factors leads to a unique result.

Similarly, it is seen that $693=21\cdot 33=9\cdot 77$, where 21, 33, 9, 77 are irreducible in the restricted realm as is also $441=(21)^2=9\cdot 49$.

Another illustration due to Hensel[1] is of interest. Let all the natural integers be distributed into two classes. Into the class C_0 let unity and those integers enter which when factored offer an even number of prime factors, while the class C_1 is to include all those integers which when factored present an odd number of prime factors. It is seen that

$$C_0 = [1,4,6,9,10,14,15,16,21,22,24, \cdots]$$
$$C_1 = [2,3,5,7,8,11,12,13,17,18,19,20,23, \cdots].$$

Let only the integers of one of the classes, say C_0, form a fixed realm. We shall confine our attention only to integers of this realm. It is seen that $210=6\cdot 35 =10\cdot 21=14\cdot 15$ which are three products of prime integers in C_0. Denote the greatest common divisor of two integers a and b by the symbol (a, b) and observe that we may write

$$\begin{aligned} 210 &= (\ 6,10)(\ 6,21)(35,10)(35,21) = 2\cdot 3\cdot 5\cdot 7, \\ &= (\ 6,14)(\ 6,15)(35,14)(35,15) = 2\cdot 3\cdot 7\cdot 5, \\ &= (10,14)(10,15)(21,14)(21,15) = 2\cdot 5\cdot 7\cdot 3. \end{aligned}$$

Observe further that the ideals in each of the last three lines are equal, for example, $(6, 10)=(6, 14)=(10, 14)$. If then we call the integers of C_0 the real integers and those of C_1 the ideal integers, it is seen that 210 is equal to the unique product of the four ideal (Kummer) integers 2, 3, 5, 7, which do not have a real existence in C_0. Notice also that the elements of the ideals, say 6, 10 of $(6, 10)$ are numbers of the fixed realm C_0.

Consider next the factorization of 21 in the realm $R(\sqrt{-5})$, namely,

$$21 = 3\cdot 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}) = (4 + \sqrt{-5})(4 - \sqrt{-5}).$$

Integers of this realm are of the form $a+b\sqrt{-5}$, where a and b are rational integers. Clearly there is something *common* to 7 and to at least one of the factors $(1+2\sqrt{-5})$, $(1-2\sqrt{-5})$. Take the product of these factors and form the congruence

$$1 - 2^2(-5) \equiv 0 \pmod 7. \tag{1}$$

Note that $-5\equiv 3^2$ (mod. 7), so that (1) becomes $1-2^2\cdot 3^2\equiv 0$ (mod. 7).

1°. From this it is seen that $1+2\cdot 3\equiv 0$ (mod. 7). Compare this congruence with the factor $1+2\sqrt{-5}$. Kummer denoted that which is common to 7 and $1+2\sqrt{-5}$ by what I call the *Kummer factor* $\{7,\ 3\}=K_1$, while the Kummer factor $K_2=\{7,\ -3\}$ denotes what is common to 7 and $1-2\sqrt{-5}$.

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[1] Hensel, *Gedächtnissrede auf E. E. Kummer,* Abhandlungen zur Geschichte der Mathematischen Wissenschaften, vol. 29 (1910), No. 22.

Observe in a similar manner the congruence

$$(2) \qquad 1 - 2^2(-5) \equiv 0 \pmod{3}$$

and note that $-5 \equiv 2^2$ (mod. 3). Writing the congruence (2) in the form $1-2^2 \cdot 2^2 \equiv 0$ (mod. 3), it is seen that $K_3 = \{3, 2\}$ and $K_4 = \{3, -2\}$ are the Kummer factors of 3 and $1+2\sqrt{-5}$ and of 3 and $1-2\sqrt{-5}$, respectively.

2°. Similarly, that which is common to 7 and $4+\sqrt{-5}$ is the Kummer factor $K_1 = \{7, 3\}$, while $K_2 = \{7, -3\}$ is the factor of 7 and $4-\sqrt{-5}$; further, $K_3 = \{3, 2\}$ is the Kummer factor of 3 and $4+\sqrt{-5}$, $K_4 = \{3, -2\}$ being that of 3 and $4-\sqrt{-5}$.

3°. To derive that which is common to each of the pairs $1+2\sqrt{-5}$, $4+\sqrt{-5}$; $1+2\sqrt{-5}$, $4-\sqrt{-5}$; $1-2\sqrt{-5}$, $4+\sqrt{-5}$; and to $1-2\sqrt{-5}$, $4-\sqrt{-5}$, we may proceed as follows:

Denote what is common to two numbers α and β by the symbol (α, β). Thus $(\alpha, \beta) = T$, means "what is common to α and β is T." From above

$$(7,1+2\sqrt{-5}) = K_1 = (7,4+\sqrt{-5});\ (7,1-2\sqrt{-5}) = K_2 = (7,4-\sqrt{-5});$$
$$(3,4-\sqrt{-5}) = K_3 = (3,1+2\sqrt{-5});\ (3,4+\sqrt{-5}) = K_4 = (3,1-2\sqrt{-5}).$$

Observe that

$$(4+\sqrt{-5},1+2\sqrt{-5}) = (4+\sqrt{-5},1+2\sqrt{-5},2[4+\sqrt{-5}]-[1+2\sqrt{-5}])$$

so that

$$\begin{aligned}(4+\sqrt{-5},1+2\sqrt{-5}) &= (4+\sqrt{-5},1+2\sqrt{-5},7)\\ &= (4+\sqrt{-5},7) = (1+2\sqrt{-5},7) = K_1.\end{aligned}$$

Similarly,

$$(4-\sqrt{-5},1-2\sqrt{-5}) = (4-\sqrt{-5},7) = K_2 = (1-2\sqrt{-5},7).$$

Note that there is nothing save unity in common to $4+\sqrt{-5}$ and $4-\sqrt{-5}$. For, if there were, there would be something common to these two quantities and their sum, which is 8. As there is something in common with either of these quantities, say $4+\sqrt{-5}$ and 21, it would follow that there is something in common with

$$(21,8,4+\sqrt{-5}) = (21,8,8\cdot 8-3\cdot 21,4+\sqrt{-5}) = (1,4+\sqrt{-5}) = (1),$$

contrary to the hypothesis.

Next observe that

$$\begin{aligned}(4+\sqrt{-5},1-2\sqrt{-5}) &= (4+\sqrt{-5},1-2\sqrt{-5},2[4+\sqrt{-5}]\\ &\quad +[1-2\sqrt{-5}]) = (4+\sqrt{-5},1-2\sqrt{-5},9) = (4+\sqrt{-5},1\\ &\quad -2\sqrt{-5},9,[4+\sqrt{-5}][4-\sqrt{-5}]) = (4+\sqrt{-5},1-2\sqrt{-5},9,21)\\ &= (4+\sqrt{-5},1-2\sqrt{-5},3) = (4+\sqrt{-5},3) = K_4 = (1-2\sqrt{-5},3).\end{aligned}$$

Similarly,

$$(4-\sqrt{-5},1+2\sqrt{-5}) = K_3 = (4-\sqrt{-5},3) = (1+2\sqrt{-5},3).$$

It follows from the above scheme that associated with the factors

$$(K_1K_2)(K_3K_4) = (K_1K_4)(K_2K_3) = (K_1K_3)(K_2K_4)$$

are the integers

$$7\cdot 3 = (1+2\sqrt{-5})(1-2\sqrt{-5}) = (4+\sqrt{-5})(4-\sqrt{-5}).$$

Thus the unique factorization of 21 in $R(\sqrt{-5})$ is $K_1K_2K_3K_4$. In the realm of natural numbers the Kummer factors have no objective reality. Hence the name *ideal.*

In the more general[1] quadratic realm $R(\sqrt{m})$, $m \not\equiv 1$ (mod. 4), we have to do with the factorization of integers $x+y\sqrt{m}$. And as above we are led to the consideration of the congruence $x^2-my^2\equiv 0$ (mod. p). If $w^2\equiv m$ (mod. p), or $w^2-pr=m$, r an integer, we have a Kummer factor $\{p, w\}$ defined through the congruence $x+wy\equiv 0$ (mod. p). This congruence put in the form of an equation is $x=pz-wy$. From this it follows that $x^2-my^2=p(pz^2-2wzy+ry^2)$. Hence, corresponding to the Kummer factor $\{p, w\}$ of p and $x+y\sqrt{m}$, there is an associated quadratic form (p,w,r), and consequently there exists a class of equivalent forms with determinant m through which p (connected with w as above defined) may be expressed. Then and only then, when the class to which (p, w, r) belongs is a principal class $(1, 0, m)$, can p be expressed through the form $p=x^2-my^2$ $=(x+y\sqrt{m})\ (x-y\sqrt{m})$. In this case and only in this case are the Kummer ideal factors $\{p, w\}$ and $\{p, -w\}$ numbers (algebraic) and have a real existence.

We thus have the condition under which the rules of division that exist in the rational realm are also true in the quadratic realms without the necessity of introducing the Kummer factors, this being evidently the case when the number of non-equivalent classes of quadratic forms with determinant m is unity. When the quadratic form through which the prime ideal p may be expressed is not equivalent to the principal form, it is necessary to introduce the ideal factors to effect uniquely the factorization of the rational prime integers into irreducible factors. Thus it is shown that the theory of quadratic forms with determinant m is exactly correlated with the theory of complex numbers of the realm $R(\sqrt{m})$.

The Kummer theory may with some modification be so changed that the ideal factors of unreal existence may be replaced by ideals of a concrete form. For, if a Kummer ideal prime factor $\{p, w\}$ of p is defined through the congruence $x+wy\equiv 0$ (mod. p) it is seen that the collectivity (complex) of all

[1] See Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, p. 146.

integers of the form $x+y\sqrt{m}$, which are divisible by $\{p, w\}$ with a suitable choice of x, y, may also be expressed through $x=pz-wy$. And that is, the complex of all those algebraic numbers that are divisible by $\{p, w\}$ is of the form $pz+(\sqrt{m}-w)y$. This complex of numbers constitutes what is known as the modul $\mathfrak{a}=[p, \sqrt{m}-w]$. And it is further seen that any number of this modul $pz+(\sqrt{m}-w)y$ when multiplied by any integer of the realm, say $x'+y'\sqrt{m}$ is equal to

$$p(zx' - ryy' + wzy') + (\sqrt{m} - w)(yx' + pzy' - wyy'),$$

if we write $w^2 - m = pr$. Observe that this latter expression is of the form

$$pZ + (\sqrt{m} - w)Y\ ;$$

and that is, a number of the modul $\mathfrak{a}$ when multiplied by an integer of the realm $R(\sqrt{m})$ is a number of the modul $\mathfrak{a}$. The counterpart of this in the theory of rational integers is: if an integer is divisible by the rational integer a, then the product of the first integer by any other integer is divisible by a. This might in a measure be used to define a rational integer a. It is used by Dedekind to define an ideal $=\mathfrak{i}=[\alpha, \beta]$, where the element p above is replaced by α and where β stands for $\sqrt{m}-w$. Accordingly, the ideal $=\mathfrak{i}$ is defined as the complex of integers $\alpha\lambda+\beta\mu$ where λ and μ run through all the integers of the given realm, and where α and β are definite fixed integers of this realm.

Kummer, in Crelle's Journal, vol. 35 (1846), p. 359, writes as follows regarding the analogy that exists between the theory of complex (algebraic) numbers and chemistry: "Multiplication for the complex numbers corresponds to the chemical compound (Verbindung); to the chemical elements or atomic weights there correspond the prime factors, and the chemical formulae for the decomposition of bodies are precisely the same as the formulae for the factorization of numbers. And the ideal numbers of our theory appear also in chemistry as hypothetical radicals (Radicale) which have not as yet been isolated (dargestellt), but have their reality, as do the ideal numbers in their combination (Zusammensetzung). For example, fluorine[1] which has not yet been isolated (1846) and is still counted as one of the elements, is an analog of an ideal prime factor. Idealism in chemistry however is essentially different from the idealism of complex (algebraic) numbers in that chemical ideal materials when combined with actual materials produce actual compounds: but this is not the case with ideal numbers. Furthermore, in chemistry the materials making up an unknown dissolved (aufgeloesten) body may be tested by means of reagents, which produce precipitates, from which the presence of the various materials may be recognized. Observe that the multiplication of prime ideal numbers produce

[1] Fluorine was isolated in 1886 by Moissant at Paris.

a rational prime integer, so that the reagents of chemistry are the analogues of the prime ideals, these prime ideals being exactly the same as the insoluble precipitate which, after the application of the reagent, settles.

"Also, the conception of equivalence is practically the same in chemistry as in the theory of algebraic numbers. For, as in chemistry, two weights are called equivalent if they can mutually replace each other either for the purpose of neutralization or to bring about the appearance of isomorphism. Similarly, two ideal numbers are equivalent if each of them can make a real rational number out of the same ideal number."

"These analogies which are set forth here are not to be considered as mere play things of the mind' in that chemistry as well as that portion of the number theory which is here treated have the same basic concept, namely, that of combination even though within different spheres of being. And from this it follows that those things which are related to this (principle of combination) and the given concepts which necessarily follow with it (that is, with the principle of combination) must be found in both (fields) by similar methods: The chemistry on the one hand of natural materials and on the other hand the chemistry of ideal numbers may both be regarded as the realizations (Verwirklichungen) of the concept of combination and of the concept-spheres dependent upon this (principle of combination) the former to be regarded as a physical which is bound with the accidents of external existence and therefore richer, the latter a mathematical, which in its inner necessity (nothwendigkeit) is perfectly pure, but therefore also poorer than the former."

While the above is not an exact translation, a few phrases having been added to make the meaning clearer, it gives, I believe, the full and exact import of what Kummer wished to convey.

Professor Edward Zeller, the world's authority on Greek philosophy writes:[1]

"Plato's pure mathematics is primarily a preparatory stage of Dialectic (general logic), the number with which it has to do are not ideal but mathematical numbers: not identical with ideas but intermediate between them and the things of sense. Side by side with numbers, the ideas of numbers are also spoken of, but only in the same sense that ideas generally are opposed to things."

And further:

"The more exact distinction between the two kinds of numbers is this; that the mathematical consists of homogeneous unities which can therefore be reckoned together, each with each, whereas with the ideal numbers this is not the case; consequently the former expresses merely quantitative, the latter logical determinations."

It is seen from this that in the notion handed down from Plato addition is an

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[1] *Plato and the Older Academy*. By Eduard Zeller. Translated by Alleyne and Goodwin. Page 256.

operation belonging to the mathematical numbers while with the ideal numbers this is not the case. Even with the Greeks it would appear that mathematics was an apparatus for numerical calculation as well as the philosophy of thought.

From recent articles which have to do with the historical development of chemistry and other physical sciences, which have appeared for example in *Science* I am led to believe that the assertion is justified that the lines of thought as well as the laws of thought in mathematics, the sciences and philosophy are today so closely allied with those of earlier times that they are one and the same. Evolution of the human mind has been exceedingly slow. And it is gratifying to note that progress in the entire intellectual structure has been made and is being made in the right direction and along lines where the truth seems of itself to be the guide.

I cannot refrain however from noting certain difficulties which the philosophers seem to experience regarding the *Ideas* and *Numbers* of Plato:

Jowett, in *The Dialogues of Plato*, vol. 2, p. 13 (third edition), writes; "Plato's doctrine of ideas has attained an imaginary clearness and definiteness which is not to be found in his own writings." Jowett then proceeds to enumerate the ways in which these ideas are interpreted and describes the reasons of their misinterpretations and misunderstanding.

Professor Paul Shorey, in *The Unity of Plato's Thought*, Chicago Decennial Publications, first series, vol. 6, p. 82, writes: "Aristotle's account of Plato's later identification of ideas and numbers has been generally accepted since Trendelenburg's dissertation on the subject. Zeller rightly points out that the doctrine is not found in the extant writings, but adds that for Plato numbers are entities intermediate between ideas and things of sense. In my discussion of the subject I tried to establish two points; first that we need not accept the testimony of Aristotle, who often misunderstood Plato and was himself not clear as to the relation of mathematical and other ideas: second, that the doctrine of numbers as intermediate entities is not to be found in Plato, but that the passages which misled Zeller may well have been the source of the whole tradition about ideas and numbers."

It does not seem to me in view of the above quoted passages from Zeller and from what I have given above regarding the Kummer ideal numbers, that there is any incorrectness either in Zeller's interpretation or in that of the equally eminent Plato scholar. In the quoted passage, Zeller does not use the word "entity." Further Shorey (p. 83) writes; "The 'mathematical' numbers then are plainly the abstract, ideal numbers of the philosopher." Surely Kummer and Dedekind were philosophers as well as mathematicians. And, continuing, Professor Shorey writes; "The numbers of the vulgar are concrete numbered things." And this is precisely what I have indicated above. See also Shorey, *Ideas and Numbers Again*, Classical Philology, vol. 22 (1927). What particularly impresses me is the following, also from Professor Shorey (Proceedings of the Sixth International Congress of Philosophy, p. 578); "And the more closely and critically one studies Plato, the more apparent it becomes that the one way to misunderstand him is to make condescending allowance for the naiveté, the immaturity, the primitive, unscientific, unsophisticated quality of his thought. Mankind has undergone many vicissitudes of experience since Plato wrote, by which philosophers must endeavor to profit. But philosophy itself, metaphysics, epistemology, speculation about ultimates in cosmogony, religion, ethics and politics, and the subtlety and logic of the art of debate have progressed since Plato just as much as epic has improved since Homer, tragedy since Sophocles, sculpture since Phidias, and eloquence since Demosthenes." Happily Professor Shorey does not include the mathematicians in the category of those whose progress by implication has been in the negative direction.

FORCED VIBRATIONS WITH SMALL RESISTANCE, APPROXIMATING RESONANCE[1]

L. W. BLAU, University of Texas

1. Introduction. The general subject of forced vibrations has been discussed rather fully by different authors[2], but the relations derived do not adapt themselves readily to the investigation of certain especially important cases. Some of these are; 1. Approximating resonance, with small resistance; 2. Resonance, with small resistance; 3. Approximating resonance, with zero resistance. In particular, the relation of these three cases to a fourth case; 4. Resonance with zero resistance, has not been clearly set forth. Rayleigh[3] states, in discussing the equation derived by him, that "the change of phase from complete agreement to complete disagreement, which is gradual when friction acts, here" (that is, when there is resonance and the friction is permitted to approach zero) "takes place abruptly. At the same time the amplitude becomes infinite." This statement is indefinite. "At the same time" does not mean *instantaneously*, for it is well known that on solving the differential equation of case 4 the amplitude, that is the maximum of the absolute value of the displacement, does not increase to infinity monotonically, but the relative maxima of this value increase directly as the first

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[1] Read before the Texas section of the Mathematical Association of America, Jan. 28, 1928.

[2] Rayleigh, *Theory of Sound*, 2nd edition. The Macmillan company, 1894. Duffing, *Erzwungene Schwingungen Bei Veränderlicher Eigenfrequenz und Ihre Technische Bedeutung*, F. Vieweg u. Sohn, Braunschweig, 1918. Riemann-Weber, *Die Differential-und Integralgleichungen der Mechanik und Physik*, Zweiter, Physikalischer, Teil. F. Vieweg u. Sohn, Braunschweig, 1927, pp. 86 ff.

[3] *Theory of Sound*, 2nd edition. The Macmillan company, 1894, vol. 1, p. 48.