NEIGHBORLY AND CYCLIC POLYTOPES

BY

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Introduction. Let S be a finite set of points in n-space. A pair of points p and q of S are called *neighbors* if the segment joining them is an edge of the convex polytope spanned by S. Some years ago [1], I was concerned with the question of when a set S can have the property that every two of its points are neighbors. It is intuitively clear and easily proved that such "neighborly" sets in 3-space can have at most 4 points. However, in 4-space it turned out that there are neighborly sets having any finite number of points.

To be somewhat more general and also more precise, let S be a finite set in n-space.

DEFINITIONS. (1) A subset R of S is called a *set of neighbors* if there is a supporting hyperplane to S which meets S in R. (Equivalently, R spans a face of the convex hull of S.)

(2) The set S is called *m-neighborly* if every subset of m points are neighbors.

The principal result of [1] asserts that in $2m$-space there are m-neighborly sets of arbitrary finite cardinality. The proof given was somewhat lengthy and indirect. After publication of [1] Professor Coxeter suggested that explicit examples of those m-neighborly polytopes were perhaps the so-called Petrie polytopes [2] (we show this to be the case in § 3). An even simpler example of m-neighborly polytopes can be obtained from the "moment curve" θ in $2m$-space given by

$$\theta(t) = (t, t^2, \cdots, t^{2m}) . \tag{1}$$

In the next section we shall show that any n points on the moment curve form an m-neighborly set. This fact seems to have been known to many people and probably to Carathéodory as far back as 1911 as indicated in [3]. However, the short proof of neighborliness given in the next section has not to my knowledge been published previously.

The polytopes obtained as the convex hull of points on the moment curve will be called *cyclic polytopes* which are a natural generalization of convex polygons in the plane. We completely describe the face structure of these polytopes and compute the number of their $2m$-faces. This has connections with one of the main open problems in the theory of polytopes, namely, that of determining the maximum number of faces a polytope with n vertices in m-space can have. It has been conjectured that the cyclic polytopes achieve this maximum.

§ 2 is concerned with a conjecture of Motzkin that any neighborly polytope has the same face structure as a cyclic polytope. This can be shown for polytopes with fewer than $2m+4$ vertices but there seems to be some difficulty in the general case. In the final section we show that the Petrie polytopes are neighborly and that they are natural generalizations of the regular polygons in the plane.

1. The cyclic polytopes. Let $t_1, \cdots, t_n$ be n distinct real numbers and let θ_i be the $2m$-vector $\theta(t_i)$ given by (1) above.

THEOREM 1. *The set $S = \{\theta_1, \theta_2, \cdots, \theta_n\}$ is m-neighborly.*

PROOF. Given any m points in S, say $\theta_1, \cdots, \theta_m$, we will find a hyperplane through them which supports the moment curve θ, that is, we will find a $2m$-vector b and a number β_0 such that

$$\begin{aligned} &b \cdot \theta_i + \beta_0 = 0, \qquad && i = 1, \cdots, m, \\ &b \cdot \theta(t) + \beta_0 > 0 \qquad && \text{for} \quad t \neq t_i, i = 1, \cdots, m . \end{aligned} \tag{2}$$

Let $p(t)$ be the polynomial defined by

$$p(t) = \prod_{i=1}^{m} (t - t_i)^2 = \beta_0 + \beta_1 t + \cdots + t^{2m} . \tag{3}$$

Clearly $p(t)$ is non-negative and vanishes only for $t = t_i$, $i = 1, \cdots, m$. Letting $b = (\beta_1, \beta_2, \cdots, \beta_{2m-1}, 1)$ it is clear from (3) that b and β_0 satisfy (2) as required.

We shall call a polytope of the type given above a *cyclic polytope* and we proceed to describe the face structure of such polytopes. Our first result applies not only to cyclic but to any neighborly polytope.

THEOREM 2. *The points of an m-neighborly set S in $2m$-space are in general position.*

PROOF. Suppose some set R of $2m+1$ points of S lies in a hyperplane. The set R is m-neighborly since S is, but R would then be an m-neighborly set in a space of dimension $2m-1$ and as proved in [1, §3], such a set can contain at most $2m$ points, giving a contradiction.

COROLLARY. *The faces of a neighborly polytope are simplexes.*

We shall next determine exactly when a set of vertices of a cyclic polytope spans a $2m$-face. For this purpose it is convenient to assume the numbers t_i occur in increasing order so that the points $\theta_1, \theta_2, \cdots, \theta_n$ occur in order on the moment curve. Let R be any subset of S containing $2m$ points.

THEOREM 3. *The points of R are neighbors if and only if*

(3) *for any two points θ_i, θ_j not in R there are an even number of points of R between θ_i and θ_j.*

PROOF. Let $R = \{\theta_{i_1}, \cdots, \theta_{i_{2m}}\}$. We consider the polynomial

$$p(t) = \prod_{r=1}^{2m} (t - t_{i_r}) = \sum_{r=0}^{2m} \beta_r t^r . \tag{4}$$

Letting $b = (\beta_1, \cdots, \beta_{2m})$ it is clear as in Theorem 1 that the hyperplane through R is given by the equation

$$b \cdot x + \beta_0 = 0$$

and this hyperplane will support S if and only if the numbers $b \cdot \theta_i + \beta_0 = p(t_i)$ are all of the same sign for θ_i not in R. Suppose θ_i is not in R and $p(t_i)$ is positive (negative). This means from (4) that there are an even (odd) number of roots, t_{i_r}, greater than t_i. But if θ_j is also not in R then there will be an even (odd) number of roots, t_{i_r}, greater than t_j if and only if there are an even number of these roots between t_i and t_j which is precisely the condition in the statement of the theorem.

There is a simple schematic representation to illustrate Theorem 3. Imagine a set of n beads strung on a circular string and suppose $2m$ of them are black and the rest white. The black beads will then correspond to a face of S if in this necklace they occur in intervals of even length.

It is clear from Theorem 3 that all cyclic polytopes in $2m$-space having the same number of vertices are combinatorially isomorphic, that is, there is a one-one correspondence between vertices which induces a one-one correspondence between faces. Motzkin has raised the question as to whether every m-neighborly set in $2m$-space is combinatorially equivalent to a cyclic polytope. He indicated an approach to this problem which yields a partial result to be taken up in the next section. We conclude the present section by computing the number of $(2m-1)$-faces of the cyclic polytopes.

THEOREM 4. *The number of $(2m-1)$-faces of a cyclic polytope with n vertices in $2m$-space is given by the formula*

$$\binom{n-m}{m} + \binom{n-m-1}{m-1}. \tag{5}$$

PROOF. (We assume, of course, that $n > 2m$.) Let R be a face of S and let $t_{i_1} < t_{i_2} < \cdots < t_{i_{n-2m}}$ correspond to the points of S which are not in R. We distinguish two cases.

Case 1. i_1 is even. Then the roots t_i of (4) are divided into $n-2m+1$ intervals, those less than t_{i_1}, those between t_{i_1} and $t_{i_2}, \cdots$, those greater than $t_{i_{n-2m}}$. We thus have $n-2m+1$ even integers whose sum is $2m$, or dividing by 2, we have $n-2m+1$ integers whose sum is m. Conversely any such set of numbers will yield a face. But the well-known formula for the number of ways of writing a non-negative integer a as the sum of non-negative integers gives

$$\binom{(n-2m+1)+m-1}{(n-2m+1)-1} = \binom{n-m}{n-2m} = \binom{n-m}{m}.$$

Case 2. i_1 is odd. It then follows that R is divided into $n-2m+1$ intervals the first and last of which contain an odd number of elements of R while the rest contain an even number. Thus θ_1 and θ_n belong to R. If we throw θ_1 and θ_n away we are back in case 1 again where n has been reduced by 2 and m has been reduced by 1: The formula thus becomes

$$\binom{n-m-1}{m-1}$$

and adding this to the expression in case 1 we get the desired formula.

REMARK 1. As already noted, it has been conjectured that (5) gives the maximum number of faces for any polytope with n vertices in $2m$-space.

REMARK 2. Mr. Martin Fieldhouse in an unpublished paper has shown that any m-neighborly polytope in $2m$-space must have the number of $(2m-1)$-faces given by (5), and more generally he has computed the number of faces in all dimensions just from knowing that the polytope is m-neighborly. This may be a further indication that all neighborly polytopes are cyclic.

2. Neighborly polytopes which are cyclic. In this section we shall understand by a cyclic polytope one which is combinatorially isomorphic to a polytope described in the previous section. In other words, the set of points S in $2m$-space spans a cyclic polytope if its points can be cyclically ordered in such a way that a subset of $2m$ points are neighbors if and only if they form a collection of disjoint "even intervals" in this ordering, as described in §1. We shall show that if S is m-neighborly and has fewer than $2m+4$ vertices then it is cyclic.

The case $n = 2m+1$ is trivial for in this case we have a $2m$-simplex and if the vertices are ordered in any manner whatever the "evenness condition" of Theorem 3 will be satisfied for all $2m$ element subsets.

To treat the other cases we shall need the following general fact about convex polytopes.

LEMMA 1. *Let $S = \{a_1, \cdots, a_n\}$ be a finite point set in general position in m-space. For $k \leqq m$ a set R of k points are neighbors if and only if the equations*

$$\sum \lambda_i a_i = 0 , \tag{6}$$

$$\sum \lambda_i = 0 \tag{7}$$

have no nonzero solution with $\lambda_i \geqq 0$ for $a_i \in S - R$.

The necessity of the condition is clear for if R is a set of neighbors then for some vector b and number β_0

$$b \cdot a_i + \beta_0 \geqq 0 \quad \text{for } a_i \text{ in } S$$

with equality only for a_i in R. Taking scalar product of (6) with b, multiplying (7) by β_0 and adding we would then have a positive quantity equal to zero, a contradiction.

The proof of sufficiency requires some fairly standard juggling with linear inequalities. For details see [1, Lemma 1].

COROLLARY. *The set S is k-neighborly if and only if every nonzero solution of (6), (7) has at least $k+1$ positive and $k+1$ negative coordinates.*

PROOF. Suppose there is a solution with only v negative coordinates, say $\lambda_1, \cdots, \lambda_v$, where $v \leqq k \leqq m$; then $a_1, \cdots, a_v$ are not neighbors from the lemma. If there is a solution with k or fewer positive coordinates then its negative is a solution with k or fewer negative coordinates and the same argument applies.

We now consider the case of $2m+2$ points $a_1, \cdots, a_{2m+2}$ in $2m$-space. Equations (6), (7) become a system of $2m+1$ homogeneous equations in $2m$ unknowns. If S is m-neighborly then every solution of (6), (7) has exactly $m+1$ positive and $m+1$ negative coordinates and must therefore be unique up to multiplication by a scalar (otherwise we could produce a solution with at least one coordinate zero). Picking a particular solution $(\lambda_1, \cdots, \lambda_n)$ of (6), (7) we call a_i positive or negative according as λ_i is positive or negative. Now order the a_i in any manner as long as the positive and negative terms alternate in the ordering. When will $2m$ points span a face? From the lemma this will occur precisely when the remaining two points have opposite signs, but because of the way the points were ordered this will happen only when the $2m$ points in question satisfy the evenness condition of Theorem 3.

The case $n = 2m+3$ is somewhat more complicated. Let us denote by Λ the set of all solutions of (6), (7). Since we have $2m+1$ equations in $2m+3$ unknowns Λ has dimension at least 2, and again every vector in Λ has at least $m+1$ positive and $m+1$ negative coordinates. It follows that Λ cannot contain more than two independent vectors or we could form a linear combination with two coordinates zero.

Let $b = (\beta_1, \cdots, \beta_n)$ and $c = (\gamma_1, \cdots, \gamma_n)$ be a basis for Λ. We may choose c so that $\gamma_i \neq 0$ for all i and $\gamma_1 > 0$. We now reorder the vectors a_i so that

$$(8) \qquad \beta_1/\gamma_1 < \beta_2/\gamma_2 < \cdots < \beta_n/\gamma_n$$

and this will turn out to be the required ordering. The fact that all ratios in (8) are distinct follows again from the fact that no vector in Λ can have more than one zero coordinate.

We must now show that the faces of S satisfy the evenness condition with respect to the ordering given by (8). This is done by describing the "sign configuration" of the vectors in Λ, that is, the order in which positive and negative coordinates occur in these vectors. We have already noted that at most one coordinate of a vector in Λ can vanish. Further for each coordinate there is a vector in Λ vanishing at that coordinate and this vector is uniquely determined up to a scalar multiple, for if not we could again produce a vector with two zero coordinates. Calling a vector with a zero coordinate a *singular* vector we will show

LEMMA 2. *The singular vectors of Λ have coordinates which alternate in sign, i.e., deleting the zero coordinate leaves a vector whose adjacent coordinates have opposite signs.*

Assuming the lemma for the moment let us establish that S satisfies the evenness condition. First, knowing the sign configuration of the singular vectors we easily obtain the sign configuration for all vectors of Λ since any such vector can be obtained by "perturbing" a singular vector in such a way that none of the nonzero coordinates changes sign. It then follows from Lemma 2 that any vector in Λ has coordinates which alternate in sign except for one pair of adjacent coordinates which have the same sign. It is also clear that all possible sign configurations satisfying this condition can be realized by suitably perturbing the appropriate singular vector.

It is now easy to verify the evenness condition. From Lemma 1 we know that given any three points $a_{i_1}, a_{i_2}, a_{i_3}$ the remaining points will be neighbors provided that no vector in Λ has $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3} \geqq 0$. Now if the subscripts i_1, i_2, i_3 satisfy the evenness condition then the corresponding coordinates cannot all be positive because from what we have seen concerning sign configurations at least one pair, say λ_{i_1} and λ_{i_2}, are end points of an even interval on which coordinates alternate in sign. If i_1, i_2 and i_3 do not satisfy the evenness condition then two of the three intervals into which the a_i are divided will contain an odd number of points. We can then find a vector in Λ with $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3} \geqq 0$, for if, say, the interval from i_1 to i_2 is even we choose a vector in Λ having adjacent coordinates of the same sign for some pair on this interval. It follows that λ_{i_1} and λ_{i_2} will have the same sign, as will λ_{i_3} since the interval i_2, i_3 is odd. Hence by Lemma 1 again the complement of the set $\{a_{i_1}, a_{i_2}, a_{i_3}\}$ will not be a face.

Proof of Lemma 2. Let λ be a real number and define the vector

$$d(\lambda) = b - \lambda c = (\delta_1(\lambda), \cdots, \delta_n(\lambda)).$$

For λ small, in fact for $\lambda < \beta_1/\gamma_1$, $d(\lambda)$ will have the same sign configuration as c. As λ increases the coordinates of $d(\lambda)$ will change sign successively as λ passes through the points $\beta_1/\gamma_1, \beta_2/\gamma_2$, etc. Since γ_1 is positive it follows that $\delta_1(\lambda)$ will change from positive to negative as λ passes through β_1/γ_1. Next λ passes through β_2/γ_2 and $\delta_2(\lambda)$ changes sign, necessarily from negative to positive for if it also changed from positive to negative we would have encountered two successive sign changes from positive to negative. But since we started out with at most $m + 2$ positive coordinates we would then be reduced to m positive coordinates which from the corollary to Lemma 1 cannot happen for a neighborly polytope. Thus $\delta_2(\lambda)$ changes from negative to positive as λ passes through β_2/γ_2 and by the same argument $\delta_3(\lambda)$ goes from positive to negative as λ passes through β_3/γ_3, and so on. This argument shows that at each stage the coordinates of $d(\lambda)$ alternate in sign except that if $\beta_i/\gamma_i < \lambda < \beta_{i+1}/\gamma_{i+1}$ then $\delta_i(\lambda)$ and $\delta_{i+1}(\lambda)$ have the same sign. In particular if $\lambda = \beta_i/\gamma_i$ then $\delta_i(\lambda) = 0$ and we verify our assertion about the singular vectors.

3. The regular cyclic polytopes. We have noted that the cyclic polytopes in $2m$-space are in various ways generalizations of the convex polygons in the plane. In this section we shall show that there are cyclic polytopes which generalize the regular polygons.

We define a regular n-gon P_n in the plane as follows: Let R be a rotation of the plane through an angle $\theta = 2\pi/n$, i.e., R is given by the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Let e be the unit vector $(1, 0)$. Then the vertices of P_n are the vectors $e, eR, eR^2, \cdots, eR^{n-1}$, or in coordinate form $(1, 0)$, $(\cos\theta, \sin\theta)$, $(\cos 2\theta, \sin 2\theta)$, $\cdots$, $(\cos(n-1)\theta, \sin(n-1)\theta)$. It is clear that P_n is carried onto itself by the cyclic group generated by the rotation R.

We now define an analogous notion in $2m$-space. Again letting $\theta = 2\pi/n$ we define the rotation R by the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & & & & & \\ -\sin\theta & \cos\theta & & & & & \\ & & \cos 2\theta & \sin 2\theta & & & \\ & & -\sin 2\theta & \cos 2\theta & & & \\ \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & \cos m\theta & \sin m\theta \\ & & & & & -\sin m\theta & \cos m\theta \end{pmatrix}.$$

One easily verifies that the matrix is orthogonal and has order n. Let e be the vector $(1, 0, 1, 0, \cdots 1, 0)$ in $2m$-space and let P_n be the polytope spanned by $e, eR, \cdots, eR^{n-1}$. A simple inductive verification will show that

$$eR^k = (\cos k\theta, \sin k\theta, \cos 2k\theta, \sin 2k\theta, \cdots, \cos mk\theta, \sin mk\theta)\,. \tag{4}$$

It is again clear that P_n is carried into itself by the cyclic group of rigid motions generated by R. We shall now show

THEOREM 4. *P_n is m-neighborly.*

PROOF. We must find a supporting hyperplane to P_n passing through any m of its vertices. We shall as in §1 prove something more general. Consider the curve φ in $2m$-space defined by

$$\varphi(t) = (\cos t, \sin t, \cos 2t, \sin 2t, \cdots, \cos mt, \sin mt)$$

and let $t_1, t_2, \cdots, t_m$ be any m-numbers on $[0, 2\pi]$. Then there is a supporting hyperplane to φ meeting φ in exactly the points $\varphi(t_1), \varphi(t_2), \cdots, \varphi(t_m)$. To see this we consider the trigonometric polynomial

$$p(t) = \prod_{k=1}^{m} (1 - \cos(t - t_k))\,;$$

clearly $p(t) \geqq 0$ and equality holds only if $t = t_k, k = 1, \cdots, m$. But by elementary trigonometric algebra

$$p(t) = \sum_{k=1}^{m} (\alpha_k \cos kt + \beta_k \sin kt) + \alpha_0$$

for suitable numbers α_k and β_k.

It then follows that the hyperplane with the desired properties is defined by

$$a \cdot x + \alpha_0 = 0 \qquad \text{where } a = (\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_m, \beta_m)\,.$$

It is also possible to show by an argument similar to that of Theorem 3 that the polytope defined here is cyclic. We omit the proof.

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